The Integration Algorithm for Nilpotent Orbits of $\mathrm{G}/\mathrm{H}^{\star}$ Lax systems: *i.e.* for Extremal Black Holes[†]

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Abstract

Hereby we complete the proof of integrability of the Lax systems, based on pseudo-Riemannian coset manifolds $\mathrm{G}/\mathrm{H}^{\star}$, we recently presented in a previous paper [arXiv:0903.2559]. Supergravity spherically symmetric black hole solutions have been shown to correspond to geodesics in such manifolds and, in our previous paper, we presented the proof of Liouville integrability of such differential systems, their integration algorithm and we also discussed the orbit structure of their moduli space in terms of conserved hamiltonians. There is a singular cuspidal locus in this moduli space which needs a separate construction. This locus contains the orbits of Nilpotent Lax operators corresponding to extremal Black Holes. Here we intrinsically characterize such a locus in terms of the hamiltonians and we present the complete integration algorithm for the Nilpotent Lax operators. The algorithm is finite, requires no limit procedure and it is solely defined in terms of the initial data. For the $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(1,2)$ coset we give an exhaustive classification of all orbits, regular and singular, so providing general solutions for this case. Finally we show that our integration algorithm can be generalized to generic non-diagonalizable (in particular nilpotent) Lax matrices not necessarily associated with symmetric spaces.

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1 Introduction

As announced in the abstract, this paper completes the results of our very recent paper [1] where we discussed the classification of orbits for $\mathrm{G}/\mathrm{H}^{\star}$ Lax systems, using the conserved hamiltonians arising from the underlying Poisson structure and adapted Kodama's integration algorithm [2] to this case. For the motivations, the general setup and a more extended bibliography we refer the reader to that publication. Assuming the framework of [1] we just focus on the problem of nilpotent orbits, that have a distinct physical interest since they correspond to extremal Black Holes. The authors of [3], in parallel to our paper [1], have also independently adapted Kodama integration algorithm to the treatment of $\mathrm{G}/\mathrm{H}^{\star}$ for the case of non nilpotent initial conditions and have presented some particular solutions.

The results presented here are three:

A. The classification of exceptional orbits of Nilpotent Lax operators.

B. The explicit integration algorithm for such operators.

C. A generalized formula giving solely in terms of the initial data L_0 the integration of Lax equation for arbitrary non-diagonalizable and diagonalizable Lax matrices L, not necessarily representing symmetric spaces. This is specially useful if L_0 is non-diagonalizable (in particular nilpotent) since this is the only available integration formula. This result is presented in the appendix.

Although the presented integration algorithm is general, to illustrate it, we heavily rely on the simple $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(1,2)$ example, already employed in [1]. The same paradigmatic example is utilized to explain the classification of orbits.

2 The singular orbits of Nilpotent operators

The integration algorithm determining the solution of the first order differential equations for the tangent vector to a geodesic in a manifold $\mathrm{G}/\mathrm{H}^{\star}$ was discussed in [1] where their Liouville integrability was proved through the construction of the preserved hamiltonians, constructed with the following procedure. Given the Lax operator L, fulfilling the η-symmetry condition

$$(L\,\eta)^T = L\,\eta \tag{2.1}$$

with

$$\eta = \mathrm{diag}\,(-1,+1,...,-1,+1,+1,...+1) \;\; , \quad p \leq q \quad ; \quad p+q \,=\, \mathrm{N} \tag{2.2}$$

the complete set of p_{N} functions $\mathfrak{h}_{\alpha}$ that are involutive with respect to the Lie–Poisson bracket defined on the solvable Borel Lie algebra of $\mathrm{N}\times\mathrm{N}$ upper triangular matrices is

enumerated by an ordered pair of indices

$$\alpha = (a, b) \tag{2.3}$$

where:

$$\begin{aligned} a &= 0, ..., \left[\frac{\mathrm{N}}{2}\right] , \\ b &= 1, ..., \mathrm{N} - 2a . \end{aligned} \tag{2.4}$$

The functions $\mathfrak{h}_{ab}$ can be iteratively derived from the following relation:

$$\begin{aligned} &\det\left\{(L-\lambda)_{ij} : \ a+1 \le i \le \mathrm{N}, \ 1 \le j \le \mathrm{N}-a\right\} \\ &= \mathcal{E}_{a0}\left(\lambda^{\mathrm{N}-2a} + \sum_{b=1}^{\mathrm{N}-2a} \mathfrak{h}_{ab} \, \lambda^{N-2a-b}\right), \quad a = 0, ..., \left[\frac{\mathrm{N}}{2}\right] \end{aligned} \tag{2.5}$$

where, by definition, $\mathcal{E}_{a0}$ is the coefficient of the power $\lambda^{\mathrm{N}-2a}$. Among the conserved hamiltonians there are polynomial ones that depend only on the eigenvalues of the Lax operator and rational ones that instead depend also on the initial twisting data (the $\mathrm{H}^\star$-rotation of the Lax operator from its diagonal form in the case of real eigenvalues or its block-diagonal normal form in the case of complex eigenvalues). There are also among the $\mathfrak{h}_{ab}$ Casimir functions that have vanishing Poisson brackets with all the canonical variables.

As we extensively discussed in [1] the classification of spectral types, which corresponds with the classification of normal forms given in [4], can be mapped into the foliation of the space of Lax orbits spanned by the conserved hamiltonians of polynomial type.

For instance in the case of $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(1,2)$, where the general form of the Lax operator is given by [1]

$$L(t) = \begin{pmatrix} \frac{1}{\sqrt{2}}Y_1(t) - \frac{1}{\sqrt{6}}Y_2(t) & -\frac{1}{2}Y_3(t) & -\frac{1}{2}Y_5(t) \\ \frac{1}{2}Y_3(t) & -\frac{1}{\sqrt{2}}Y_1(t) - \frac{1}{\sqrt{6}}Y_2(t) & -\frac{1}{2}Y_4(t) \\ \frac{1}{2}Y_5(t) & -\frac{1}{2}Y_4(t) & \sqrt{\frac{2}{3}}Y_2(t) \end{pmatrix} \tag{2.6}$$

leading to the following system of differential equations:

$$\begin{aligned} -\frac{Y_3(t)^2}{\sqrt{2}} - \frac{Y_4(t)^2}{2\sqrt{2}} - \frac{Y_5(t)^2}{2\sqrt{2}} + \frac{d}{dt}\,Y_1(t) &= 0 , \\ -\frac{1}{2}\sqrt{\frac{3}{2}}Y_4(t)^2 + \frac{1}{2}\sqrt{\frac{3}{2}}Y_5(t)^2 + \frac{d}{dt}\,Y_2(t) &= 0 , \\ -\sqrt{2}Y_1(t)Y_3(t) - Y_4(t)Y_5(t) + \frac{d}{dt}\,Y_3(t) &= 0 , \\ \frac{Y_1(t)Y_4(t)}{\sqrt{2}} + \sqrt{\frac{3}{2}}Y_2(t)Y_4(t) - Y_3(t)Y_5(t) + \frac{d}{dt}\,Y_4(t) &= 0 , \\ -\frac{Y_1(t)Y_5(t)}{\sqrt{2}} + \sqrt{\frac{3}{2}}Y_2(t)Y_5(t) + \frac{d}{dt}\,Y_5(t) &= 0 , \end{aligned} \tag{2.7}$$

there are just a quadratic and a cubic hamiltonian

$$\mathfrak{h}_1 \doteq \mathfrak{h}_{01} = 0 \,, \tag{2.8}$$

$$\mathfrak{h}_2 \doteq \mathfrak{h}_{02} = \frac{1}{2}Y_1(t)^2 + \frac{1}{2}Y_2(t)^2 - \frac{1}{4}Y_3(t)^2 + \frac{1}{4}Y_4(t)^2 - \frac{1}{4}Y_5(t)^2 \,, \tag{2.9}$$

$$\begin{aligned}\mathfrak{h}_3 \doteq \mathfrak{h}_{03} = {} & \frac{Y_2(t)^3}{3\sqrt{6}} - \frac{Y_1(t)^2 Y_2(t)}{\sqrt{6}} + \frac{Y_3(t)^2 Y_2(t)}{2\sqrt{6}} + \frac{Y_4(t)^2 Y_2(t)}{4\sqrt{6}} - \frac{Y_5(t)^2 Y_2(t)}{4\sqrt{6}} \\ & - \frac{Y_1(t) Y_4(t)^2}{4\sqrt{2}} - \frac{Y_1(t) Y_5(t)^2}{4\sqrt{2}} + \frac{1}{4} Y_3(t) Y_4(t) Y_5(t)\end{aligned} \tag{2.10}$$

while the rational hamiltonian is the following one:

$$\mathfrak{h}_4 \doteq \mathfrak{h}_{11} = \frac{Y_1(t)}{\sqrt{2}} + \frac{Y_2(t)}{\sqrt{6}} - \frac{Y_3(t) Y_4(t)}{2 Y_5(t)} \,. \tag{2.11}$$

The space of orbits is separated in two distinct regions by the value of the discriminant

$$\Delta \equiv -12 \mathfrak{h}_2^3 + 81 \, \mathfrak{h}_3^2 \,. \tag{2.12}$$

- In the region where $\Delta < 0$ we have three distinct real eigenvalues.
- In the region where $\Delta > 0$ there is one real eigenvalue and a pair of complex conjugate eigenvalues.
- The locus $\Delta = 0$ corresponds to orbits admitting an enhanced symmetry, except at the cusp.

2.1 The orbits of the $\mathrm{J} = 2$ representation of $\mathrm{SO}(1,2)$

The easiest way to discuss the space of orbits for the $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(1,2)$ case is to recall that the coset generators span the five dimensional $\mathrm{J} = 2$ representation of $\mathrm{SO}(1,2)$, drawing an analogy to the well known case of orbits of the $\mathrm{J} = 1$ representation. In the latter case we deal with vectors and the space of orbits decomposes into three sectors:

1. The orbits of time-like vectors $(v\,,\,v) > 0$ which admit $\mathrm{SO}(2)$ as stability subgroup and therefore span the coset $\mathrm{SO}(1,2)/\mathrm{SO}(2)$.
2. The orbits of space-like vectors $(v\,,\,v) < 0$ which admit $\mathrm{SO}(1,1)$ as stability subgroup and therefore span the coset $\mathrm{SO}(1,2)/\mathrm{SO}(1,1)$.
3. The orbits of light-like vectors $(v\,,\,v) = 0$ which admit no stability subgroup and lie on the light-cone which separates the two regions of time-like and space-like orbits.

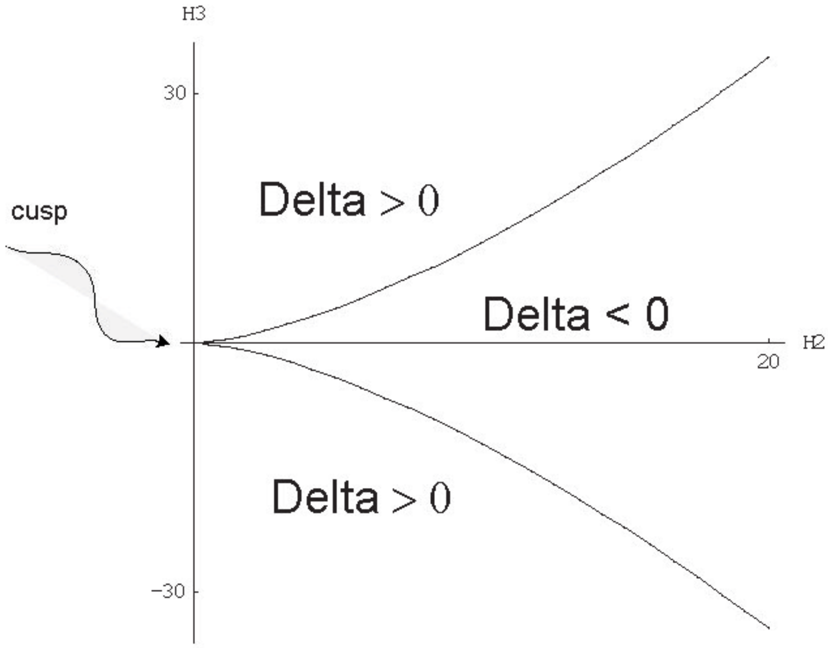

Figure 1: Orbit structure in the $\mathfrak{h}_2, \mathfrak{h}_3$ plane. The cuspidal point corresponds to the Nilpotent orbits

In the J = 2 case the role of the light-cone is played by the $\Delta = 0$ locus of vanishing discriminant. The orbits with $\Delta > 0$ have the eigenvalue structure we have just discussed and admit no stability subgroup. This can be directly verified using the generators of

$SO(1,2)$ given in [1] for the $\mathrm{J}=2$ representation and recalled here for convenience:

$$\mathrm{R}(\mathrm{J}_1) \;=\; \begin{pmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \;, \quad \mathrm{R}(\mathrm{J}_2) \;=\; \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} \;,$$

$$\mathrm{R}(\mathrm{J}_3) \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & \sqrt{3} & 0 & 0 & 0 \end{pmatrix} \;. \tag{2.13}$$

In this way the original five dimensional space is parameterized for $\Delta > 0$ orbits by the values of the two hamiltonians $\mathfrak{h}_2\,,\,\mathfrak{h}_3$, which label the orbit, and by the three parameters of $\mathrm{SO}(1,2)$ that span it. Similar conclusion one draws for the $\Delta < 0$. Apart from the difference in spectral type these orbits behave exactly in the same way. There is no stability subgroup.

On the other hand the locus $\Delta = 0$ contains 5-vectors that always admit a stabilizer $\mathcal{S}$, either belonging to the compact $\mathcal{S} \in \mathfrak{so}(2)$ subalgebra or to a non-compact subalgebra $\mathcal{S} \in \mathfrak{so}(1,1)$ of $\mathfrak{so}(1,2)$. For these orbits the discriminant Δ vanishes, yet the individual hamiltonians $\mathfrak{h}_2$ and $\mathfrak{h}_3$ are generically different from zero. In this way each of the $\Delta = 0$ orbits spans either the $\mathrm{SO}(1,2)/\mathrm{SO}(2)$ or the $\mathrm{SO}(1,2)/\mathrm{SO}(1,1)$ coset.

There is finally the cuspidal locus displayed in fig.1 where both $\mathfrak{h}_2$ and $\mathfrak{h}_3$ are zero. The five-vectors lying in this locus correspond to Nilpotent non-diagonalizable 3×3 matrices as, for example, the following:

$$\Omega = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \;. \tag{2.14}$$

This provides an intrinsic characterization of the Nilpotent orbit: vanishing of both polynomial hamiltonians.

Another interesting and equivalent algebraic characterization is the following. The 2-dimensional space of non-compact generators of the Lie algebra $\mathfrak{so}(1,2)$ given by the span of J_1 and J_3 contains operators whose eigenvalues are necessarily:

$$-\,2\,\ell\,,\;-\ell\,,\;0\,,\;+\ell\,,\;+2\,\ell \tag{2.15}$$

where ℓ is an arbitrary normalization of the operator. All the Nilpotent operators L belonging to the singular cuspidal orbit $\mathcal{C}_0$ are eigenvectors of one of these non-compact generators G with a finite eigenvalue $\mu \neq 0$:

$$L \in \mathcal{C}_0 \Leftrightarrow \forall G \in \mathfrak{so}(1,2) \; : \; [G \, , \, L] \, = \, \mu \, L \, . \tag{2.16}$$

On the contrary one can easily verify that no five-vector corresponding to regular orbits can be eigenvector of any of the $\mathfrak{so}(1,2)$ generators.

In view of these remarks we see that the $\Delta = 0$ locus is also alternatively characterized as the set of operators that are eigenstates of some $\mathfrak{so}(1,2)$ generator. Null eigenstates occur along the two branches and give orbits with enhanced symmetry (stability subgroup). Eigenstates of non-vanishing eigenvalue occur only at the cusp.

Conjecture 2.1 *For higher groups with more hamiltonians, the secular equation has typically degree higher than four and the discussion of discriminants becomes unavailable. Yet the second algebraic characterization of cuspidal orbits of Nilpotent operators that, by definition, have all vanishing eigenvalues and are not diagonalizable, remains viable and appears very promising. Indeed we conjecture that the orbits of such Lax operators can be found by this method, namely searching for eigenstates of the non-compact generators of* $H^\star$*.*

We stress that in the case of supergravity billiards [5], [6] the isotropy group H is compact and therefore there are no real Lax operators that can be eigenstates of any generator of H. It is precisely the pseudo-Riemannian nature of the coset $\mathrm{G/H}^\star$ what allows for the appearance of nilpotent Laxes.

For this singular, isolated class of initial conditions the integration algorithm was not provided in [1] since one of its ingredients consists of the diagonalization of the initial Lax. The next section fills this gap showing that the solution can be directly and simply constructed in terms of the initial Lax L_0.

3 The integration algorithm for Nilpotent Lax operators

Let us assume that the initial data for the Lax equation

$$\frac{d}{dt} L(t) + [\, L_{>} \, - \, L_{<} \, , \, L(t) \,] \, = \, 0 \tag{3.1}$$

are provided by an η-symmetric matrix:

$$L(0) \, = \, L_0 \quad ; \quad L_0 \, \eta \, = \, \eta \, L_0^T \tag{3.2}$$

which is also nilpotent and therefore not diagonalizable. Let us call n the minimal degree of nilpotency[1]:

$$L_0^n \;=\; 0\,. \tag{3.3}$$

The complete solution for $L(t)$ that admits such an initial condition is constructed in the following way. First define the following building blocks:

1. The following $\mathrm{N}\times\mathrm{N}$ matrix function:

$$\mathcal{C}(t) \;= e^{-2\,t\,L_0} \;=\; \sum_{k=0}^{n-1} \frac{1}{k!}\,(-2t)^k\;L_0^k\,. \tag{3.4}$$

2. The minor functions constructed in the following way:

$$\begin{aligned}
\mathfrak{M}_{ik}(t) &= (-1)^{i+k}\,\mathrm{Det}\left(\begin{array}{ccc} \mathcal{C}_{1,1}(t) & \dots & \mathcal{C}_{1,i-1}(t) \\ \vdots & \vdots & \vdots \\ \widehat{\mathcal{C}_{k,1}}(t) & \dots & \widehat{\mathcal{C}_{k,i-1}}(t) \\ \vdots & \vdots & \vdots \\ \mathcal{C}_{i,1}(t) & \dots & \mathcal{C}_{i,i-1}(t) \end{array}\right),\ 1\le k\le i\,;\ 2\le i\le N\,,\\
\mathfrak{M}_{11}(t) &= 1
\end{aligned} \tag{3.5}$$

where the hats on the entries corresponding to the k-th row mean that such a row has been suppressed giving rise to a squared $(i-1)\times(i-1)$ matrix of which one can calculate the determinant.

3. The determinant functions:

$$\mathfrak{D}_i(t) \;=\; \mathrm{Det}\left(\begin{array}{ccc} \mathcal{C}_{1,1}(t) & \dots & \mathcal{C}_{1,i}(t) \\ \vdots & \vdots & \vdots \\ \mathcal{C}_{i,1}(t) & \dots & \mathcal{C}_{i,i}(t) \end{array}\right)\,,\qquad \mathfrak{D}_0(t) := 1\,. \tag{3.6}$$

Then in terms of them we write the entries of the *time*-evolving Lax operator in the following way:

$$L_{pq}(t) \;=\; \frac{\eta_{qq}}{\sqrt{\mathfrak{D}_p(t)\,\mathfrak{D}_{p-1}(t)\mathfrak{D}_q(t)\,\mathfrak{D}_{q-1}(t)}}\sum_{k=1}^{p}\sum_{\ell=1}^{q}\mathfrak{M}_{pk}(t)\;\left(\mathcal{C}(t)\,L_0\,\eta\right)_{k\ell}\;\mathfrak{M}_{q\ell}(t)\,. \tag{3.7}$$

[1]In the $\mathrm{SL}(3,\mathbb{R})$ case the degree of nilpotency is either $n=2$ or $n=3$. It seems that the same occurs also in the case of more general cosets $\mathrm{G}/\mathrm{H}^\star$ occurring in supergravity theories. However the algorithm we present applies to any finite degree of nilpotency n.

As an example of the method we give the explicit form of the Lax operator which is produced by the integration with initial conditions provided by the operator Ω displayed in (2.14):

$$L_1(t) = \begin{pmatrix} \frac{2t}{1-2t^2} & -\frac{\sqrt{1+2t^2}}{1-2t^2} & 0 \\ \frac{\sqrt{1+2t^2}}{1-2t^2} & \frac{4t}{4t^4-1} & \frac{\sqrt{1-2t^2}}{1+2t^2} \\ 0 & \frac{\sqrt{1-2t^2}}{1+2t^2} & \frac{2t}{1+2t^2} \end{pmatrix} . \tag{3.8}$$

This is a representative of the orbit where the Lax operator has degree of nilpotency $n = 3$. It corresponds to the eigenvalue $\mu = 1$ of the non compact $\mathfrak{so}(1,2)$ generator $\mathrm{R}(\mathrm{J}_3)$.

It is quite interesting to employ the algorithm for the square of the matrix Ω which is an eigenvector corresponding to eigenvalue $\mu = 2$ and has degree of nilpotency $n = 2$. The result is a very simple solution with a completely different analytic behaviour

$$L_2(t) = \frac{1}{1+2t} \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} . \tag{3.9}$$

3.1 Summarizing

We conclude that the two above presented solutions are the representatives of the two nilpotent inequivalent orbits corresponding to degree of nilpotency $n = 3$ and $n = 2$, respectively. Any other solution of the same classes is obtained by applying our algorithm to an $\mathrm{SO}(1,2)$ rotation of the corresponding initial data. All in all the three-dimensional manifold of nilpotent Lax operators is provided by those matrices (2.6) whose five parameters Y_i satisfy the two constraints $\mathfrak{h}_2(Y) = \mathfrak{h}_3(Y) = 0$, according to the definitions (2.9–2.10) of the quadratic and cubic hamiltonians.

The other exceptional orbits are arranged along the two branches of the $\Delta = 0$ locus and each of them spans either an $\mathrm{SO}(1,2)/\mathrm{SO}(2)$ or an $\mathrm{SO}(1,2)/\mathrm{SO}(1,1)$ coset manifold since there is either an $\mathfrak{so}(1,1)$ or an $\mathfrak{so}(2)$ stability subalgebra. These orbits correspond to diagonalizable Lax operators and are covered by the algorithm presented in our previous paper [1].

4 Conclusions

In this note we have shown how the integration algorithm for Lax equations can be extended also to the case of non-diagonalizable (see, also the appendix) initial data. Applications to the construction of extremal BPS solutions is postponed to future publications.

Aknowledgments We would like to express our gratitude to our frequent collaborator and excellent friend Mario Trigiante for the exchange of useful information we had with

him in these last few days. He pointed out to us the relevance of the nilpotent case and formulated the problem that we were able to solve in complete generality.

Note added in revised version We recall that a particular solution corresponding to a specific nilpotent initial condition for the $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(1,2)$ case was presented in the revised version of [3] which appeared on the hep-th ArXiv the very same day as the first version of the present paper, where the problem obtained its general solution.

A A generalized formula

Although the physical or geometrical applications of such an equation are not immediately evident to us we remark that one could consider a generalization of the Lax equation for symmetric spaces $\mathrm{G}/\mathrm{H}^\star$ by writing the very same differential condition (3.1) imposed on a matrix $L(t)$ which is not required to satisfy the η-symmetry conditions (3.2). In other words we can extend Lax equation to matrices $L(t)$ that are not necessarily elements of the orthogonal complement $\mathbb{K}$ to a subalgebra $\mathbb{H}^\star \subset \mathbb{G} \subset \mathfrak{sl}(\mathrm{N},\mathbb{R})$.

It is interesting that we can integrate Lax equation in a completely general way with arbitrary initial Lax matrices L_0. This is done by substituting eq.(3.7) with the following one:

$$L_{pq}(t) \;=\; \frac{1}{\sqrt{\mathfrak{D}_p(t)\,\mathfrak{D}_{p-1}(t)\mathfrak{D}_q(t)\,\mathfrak{D}_{q-1}(t)}} \sum_{k=1}^{p}\sum_{\ell=1}^{q} \mathfrak{M}_{pk}(t)\ \left(\mathcal{C}(t)\,L_0\right)_{k\ell}\,\widetilde{\mathfrak{M}}_{q\ell}(t) \tag{A.1}$$

where

$$\begin{aligned} &\widetilde{\mathfrak{M}}_{ik}(t) := (-1)^{i+k}\,\mathrm{Det}\begin{pmatrix} \mathcal{C}_{1,1}(t) & \dots & \widehat{\mathcal{C}_{1,k}}(t) & \dots & \mathcal{C}_{1,i}(t)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathcal{C}_{i-1,1}(t) & \dots & \widehat{\mathcal{C}_{i-1,k}}(t) & \dots & \mathcal{C}_{i-1,i}(t)\end{pmatrix},\\ &1\le k\le i\ ;\ 2\le i\le \mathrm{N}\quad ;\quad \widetilde{\mathfrak{M}}_{11}(t)=1\,, \end{aligned} \tag{A.2}$$

and all other definitions remaining the same as in the main text. In eq.(A.2) the hatted k-th column is deleted just as in eq.(3.5) it was deleted the k-th row.

This expression gives the solution to Lax equation (3.1) for the case of generic initial conditions. It applies in particular to the case of diagonalizable initial matrices L_0. In that case, however, eq.(A.1) is only formal and not too useful since the function $\mathcal{C}(t)$ (3.4) is constructed by a matrix exponentiation and involves summing an infinite series. On the contrary for nilpotent matrices L_0 eq.(A.1) provides a useful explicit general integral. The class of nilpotent matrices is much larger than the class of nilpotent η-symmetric matrices (3.2), the latter being associated with symmetric spaces.

Although the physical meaning of Lax equation in the more general context of nilpotent, but not necessarily η-symmetric matrices (3.2) is so far unknown, yet it is worth

mentioning the existence of this generalized integration formula whose applications will certainly be discovered soon.

Now, let us derive another equivalent representation of equations (A.1) which will be useful in case of both nilpotent and non-nilpotent, η-symmetric and η-non-symmetric initial data L_0 entering these equations.

At first, we recall that both diagonalizable and non-diagonalizable (in particular nilpotent) generic initial conditions for the Lax matrix L_0 can be represented in the Jordan normal form

$$L_0 \;=\; \Phi(0)\,\mathcal{J}\,\Phi(0)^{-1} \tag{A.3}$$

where $\Phi(0)$ is an invertible $N \times N$ matrix, $\mathcal{J}$ is a block-diagonal $N \times N$ matrix with the $d_\alpha \times d_\alpha$ matrix sub-blocks J_{λ_α}

$$\mathcal{J} \;=\; \begin{pmatrix} \mathcal{J}_{\lambda_1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_{\lambda_m} \end{pmatrix}, \quad \mathcal{J}_{\lambda_\alpha} \;=\; \begin{pmatrix} \lambda_\alpha & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & \lambda_\alpha \end{pmatrix}, \tag{A.4}$$

$$\alpha \;=\; 1,\dots,m\,, \quad 1 \le m \,\le N\,, \quad d_\alpha \ge 1\,, \quad \sum_{\alpha=1}^{m} d_\alpha \;=\; N$$

and λ_α are constants. Substituting eq. (A.3) into equations (A.1) and (3.4), the latter can identically be represented in the following equivalent form:

$$L(t) \;=\; \Phi(t)\,\mathcal{J}\,\Phi(t)^{-1}\,, \tag{A.5}$$
$$\mathcal{C}(t) \;=\; \Phi(0)\,e^{-2t\,\mathcal{J}}\,\Phi(0)^{-1} \tag{A.6}$$

where

$$\Phi_{ij}(t) \;=\; \frac{1}{\sqrt{\mathfrak{D}_i(t)\mathfrak{D}_{i-1}(t)}}\sum_{s=1}^{N} \mathrm{Det}\begin{pmatrix} \mathcal{C}_{1,1}(t) & \dots & \mathcal{C}_{1,i-1}(t) & \Phi_{1s}(0) \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{C}_{i,1}(t) & \dots & \mathcal{C}_{i,i-1}(t) & \Phi_{is}(0) \end{pmatrix} \left(e^{-t\,\mathcal{J}}\right)_{sj}\,,$$
$$\left(\Phi^{-1}\right)_{ji}(t) \;=\; \frac{1}{\sqrt{\mathfrak{D}_i(t)\mathfrak{D}_{i-1}(t)}}\sum_{s=1}^{N} \left(e^{-t\,\mathcal{J}}\right)_{js} \mathrm{Det}\begin{pmatrix} \mathcal{C}_{1,1}(t) & \dots & \mathcal{C}_{1,i}(t) \\ \vdots & \vdots & \vdots \\ \mathcal{C}_{i-1,1}(t) & \dots & \mathcal{C}_{i-1,i}(t) \\ \left(\Phi^{-1}\right)_{s1}(0) & \dots & \left(\Phi^{-1}\right)_{si}(0) \end{pmatrix} \tag{A.7}$$

and the functions $\mathfrak{D}_i(t)$ are defined in eq. (3.6).

For the particular case of diagonalizable initial conditions, i.e. when $m = N$ ($d_\alpha = 1$, $\alpha = 1, \dots, N$), the corresponding matrix $\mathcal{J}$ (A.4) becomes a diagonal matrix of the N-eigenvalues λ_α ($\alpha = 1, \dots, N$), and equations (A.5–A.7) reproduce the general solutions to Lax equations (3.1) for the case of *generic diagonalizable* Lax matrices derived in [7] for the first time. What concerns expressions (A.5–A.7) in the case $m \neq N$, they are a generalization of the above-mentioned general solutions to the case of *generic non-diagonalizable* Lax matrices constructed, to our best knowledge, for the first time in the present paper.

When deriving eqs. (A.5–A.7) starting from eq. (A.1), we did not use any properties of the Lax operators L_0 but its rather general representation (A.3). In case of diagonal matrices $\mathcal{J}$ we proved the correctness of derived expressions (A.5–A.7) by their established relationship with the earlier known solution [7] of Lax equations (3.1), as it was already mentioned above. We also verified their correctness in case of $\frac{SL(N)}{SO(p,N-p)}$ cosets at $N = 2\,,3\,,4$ and 5 for non-diagonalizable nilpotent matrices $\mathcal{J}$. Altogether, this gives an evidence in favour of correctness of eqs. (A.5–A.7) in case of generic matrices $\mathcal{J}$ (A.3) as well.

The constructed general solutions (A.5–A.7) are explicitly parameterized by the initial data encoded in the matrices $\mathcal{J}$ and $\Phi(0)$. In order these solutions could be expressed in terms of elementary functions, the exponential $e^{-t\,\mathcal{J}}$, entering into relations (A.6) and(A.7), has to admit a closed form in terms of elementary functions. Obviously, it is a very simple task for diagonal and nilpotent matrices $\mathcal{J}$. Thus, in the latter case all eigenvalues are zero since $\mathcal{J}$ in (A.4) is a nilpotent matrix, and the expansion of the exponential $e^{-t\,\mathcal{J}}$ in power series with respect to $\mathcal{J}$ terminates at some finite order. It is more complicated but in principle a solvable task for many interesting, more generic non-nilpotent and non-diagonalizable cases.

Thus, for an example in case of η-symmetric Lax operators L_0 (3.2), $\Phi(t)$ is a pseudo-orthogonal matrix, $\Phi(t) \in SO(p,q)$, i.e. $\Phi^{-1}(t) = \eta\, \Phi^T(t)\, \eta$. In this case the non-diagonalizable matrices $\mathcal{J}$ have a very simple structure [4]:

$$\mathcal{J} \,=\, Q \,+\, Nil\,, \quad [Q\,,\, Nil] \,=\, 0 \tag{A.8}$$

where Nil is a nilpotent matrix and Q is a block-diagonal matrix with 2×2 and 1×1 sub-blocks [4, 1]. Therefore, the exponential $e^{-t\,\mathcal{J}}$ can be factorized

$$e^{-t\,\mathcal{J}} \,=\, e^{Q}\, e^{Nil} \tag{A.9}$$

and both exponentials, entering on the r.h.s. of the latter relation, can easily be expressed in terms of elementary functions. A detailed discussion of this interesting case will be presented elsewhere.

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